Persistent Homology via the Möbius Inversion

Alex McCleary

Ohio State University

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Overview

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- The incidence algebra of P has several distinguished functions.

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- This condition is equivalent to the statement that $f = \partial f * \zeta$ or $f * \mu = \partial f$.

Example

Suppose a snowstorm is hitting Columbus. Let f be the function recording the hourly snowfall totals.

Time	$f(t)$
0	0
1	4
2	6
3	6
4	8

$$\zeta = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mu = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\partial f = f * \mu = (0, 4, 2, 0, 2)$$

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- The function ∂f captures the rate of change of f.
- The Möbius inversion condition is analogous to the fundamental theorem of calculus.

Filtrations

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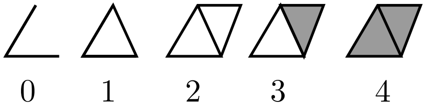

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- Let $\mathbb{I}(P)_{\supseteq}$ be the poset with the same underlying set and order $[a, b] \supseteq [c, d]$ if $a \leqslant c$ and $d \leqslant b$.

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- The *rank function* of a P-filtration F is the function $d\mathsf{F} : \mathbb{I}(P)_{\supseteq} \to \mathbb{Z}$ that assigns to any $[a, b] \in \mathbb{I}(P)_{\supseteq}$ the dimension of the image of the induced map $H_i(\mathsf{F}(a)) \to H_i(\mathsf{F}(b))$.

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- The rank function can be interpreted as the function counting the number of homological features that persist from a to b.

Birth-Death Functions

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- The *birth-death function* of a P-filtration F is the function $ZB\mathsf{F} : \mathbb{I}(P)_{\leqslant} \to \mathbb{Z}$ that assigns to an interval $[a, b] \in \mathbb{I}(P)_{\leqslant}$ the dimension of $Z(\mathsf{F}(a)) \cap B(\mathsf{F}(b))$.

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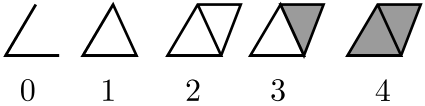
- The birth-death function can be interpreted as counting the number of homological features that were born at or before a and die at or before b.

Example

Let F be the filtration below and fix the homology dimension at 1.



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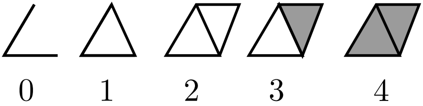

The rank function of F assigns value 1 to the intervals $[1,1], [1,2], [1,3], [2,3]$, and $[2,3]$; 2 to the interval $[2,2]$; and 0 to all other intervals.
The birth-death function of F assigns value 1 to the intervals $[1,4], [2,3]$, and $[3,3]$; 2 to the intervals $[2,4], [3,4]$, and $[4,4]$; and 0 to all other intervals.

The Persistence Diagram

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- Explicitly, we have that the persistence diagram of F is the function $\sigma : \mathbb{I}(P) \to \mathbb{Z}$ where

$$\sigma = BD\mathsf{F} * \mu_{\leqslant} = d\mathsf{F} * \mu_{\supseteq}.$$

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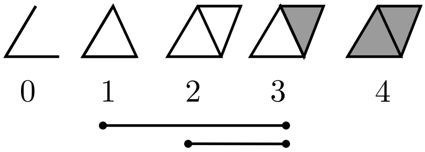
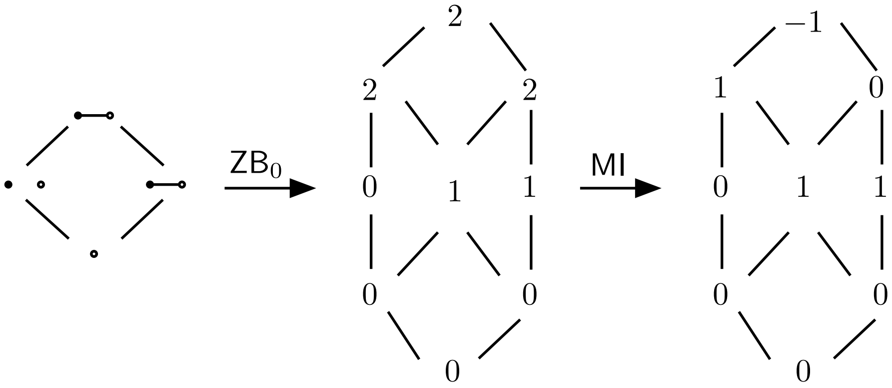


Another Example



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- These functions can easily be replaced with even more general invariants such as the generalized rank function [Kim,Mémoli].
- This approach allows us to define persistence diagrams for filtrations over more general posets.

The End